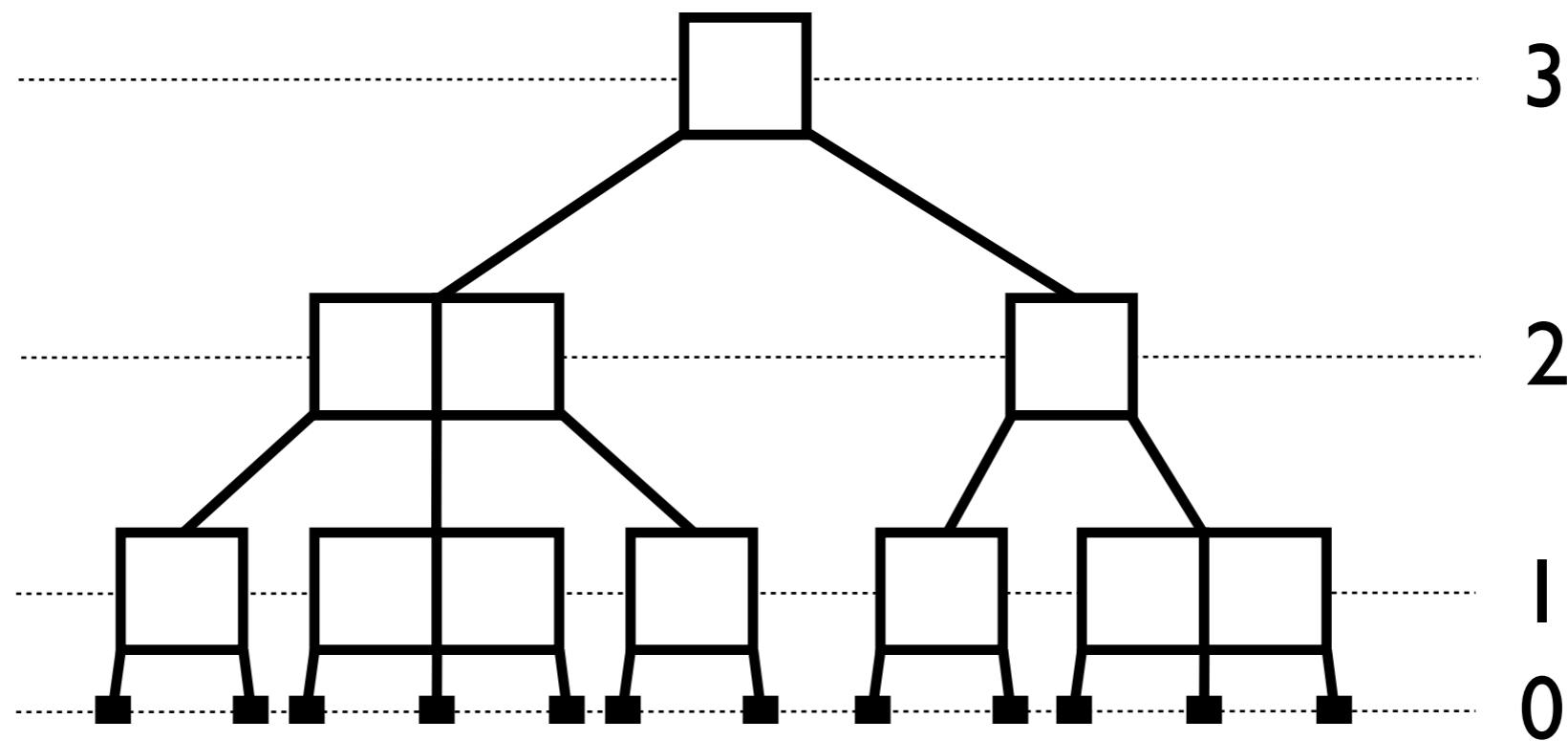


Getting data structures right

with GADTs and nested types



@mbrcknl

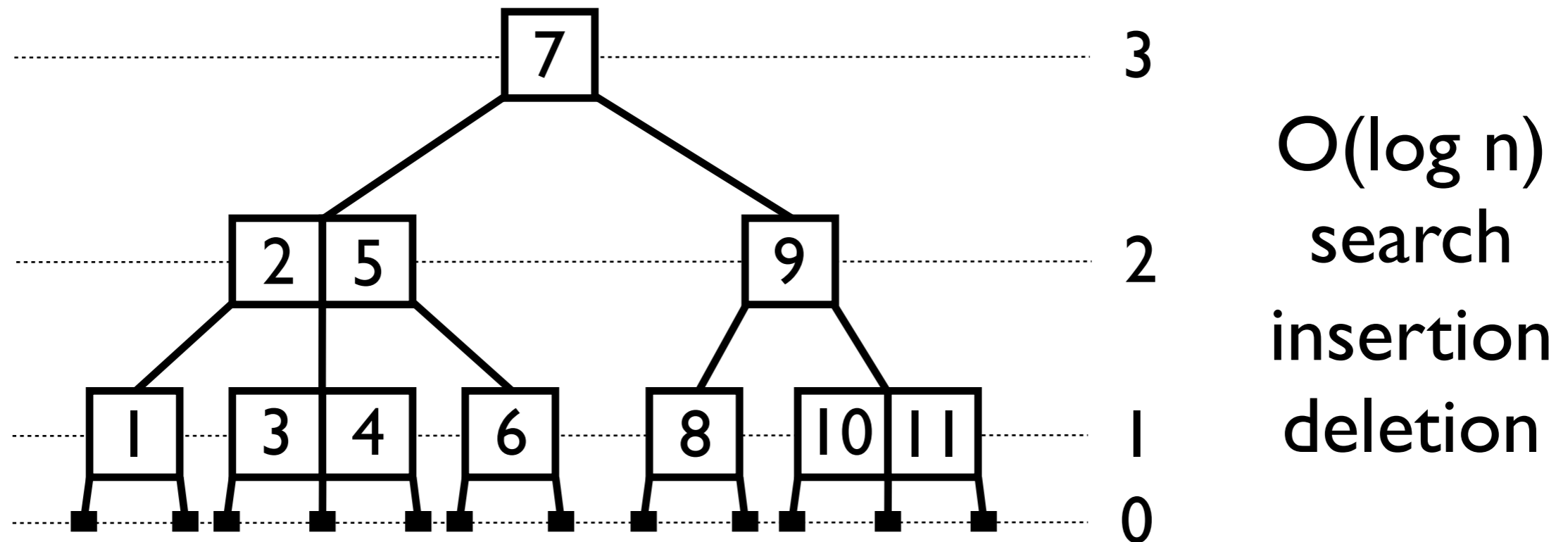
matthew.brecknell.net

2-3 B-tree comprises either:

- {one, two} data + {two, three} subtrees
- a leaf containing nothing at all

Every leaf is equidistant from the root

Data are ordered left to right



2-3 B-tree comprises either:

- {one, two} data + {two, three} subtrees
- a leaf containing nothing at all

data N a

= T1 (T a) a (T a)

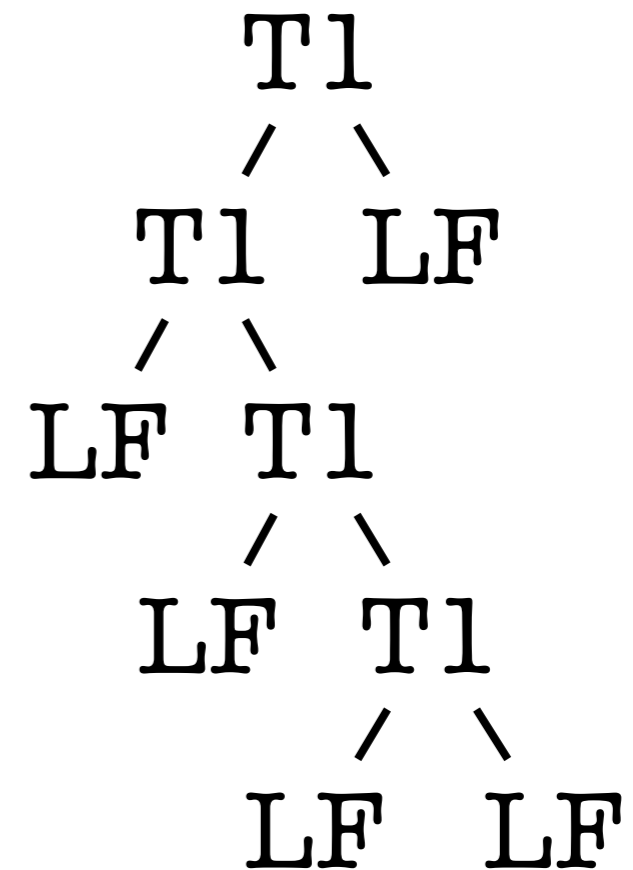
| T2 (T a) a (T a) a (T a)

data T a

= Br (N a)

| LF

$O(n)$ search



2-3 B-tree comprises either:

- {one, two} data + {two, three} subtrees
- a leaf containing nothing at all

data T a

= Br (N a)

| LF

ghci> :t Br

Br :: N a → T a

2-3 B-tree comprises either:

- {one, two} data + {two, three} subtrees
- a leaf containing nothing at all

data T a

= Br (N a)

| LF

data T a where

Br :: N a → T a

LF :: T a

2-3 B-tree comprises either:

- {one, two} data + {two, three} subtrees
- a leaf containing nothing at all

Every leaf is equidistant from the root

2-3 B-tree comprises either:

- {one, two} data + {two, three} subtrees
- a leaf containing nothing at all

Subtrees must have the same height

data Nat = Z | S Nat

data T n a where

Br :: N n a → T (S n) a

LF :: T Z a

data N n a

= T1 (T n a) a (T n a)

| T2 (T n a) a (T n a) a (T n a)

```
data Nat = Z | S Nat
```

```
data T n a where
```

```
  Br :: N n a → T (S n) a
```

```
  LF :: T Z a
```

```
data N n a
```

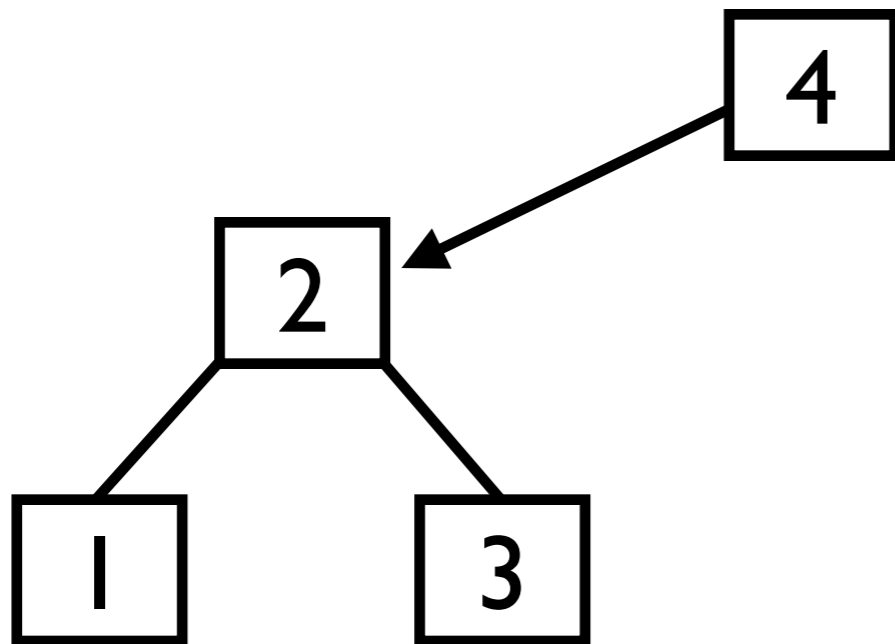
```
  = T1 (T n a) a (T n a)
```

```
  | T2 (T n a) a (T n a) a (T n a)
```

Data must be ordered left to right

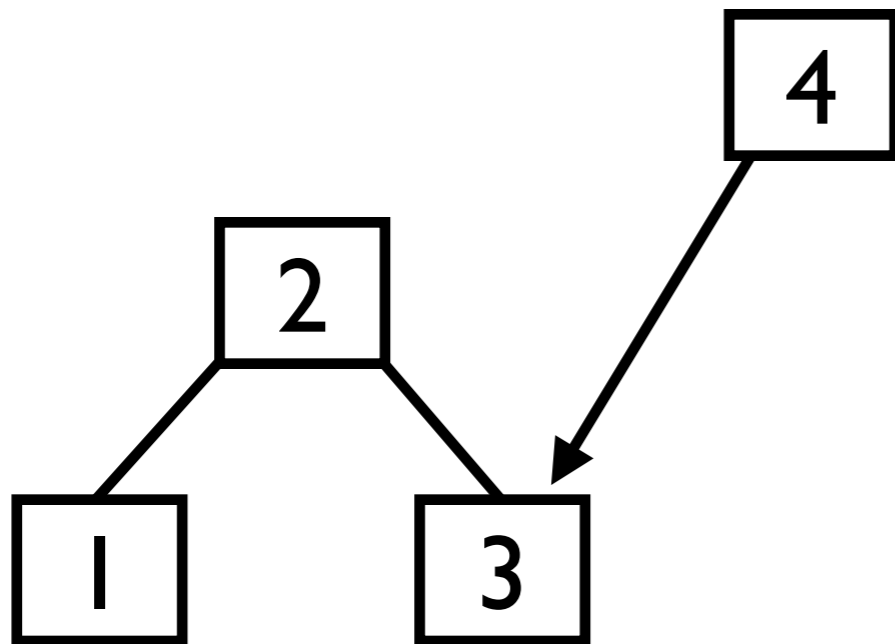
insert 4

search

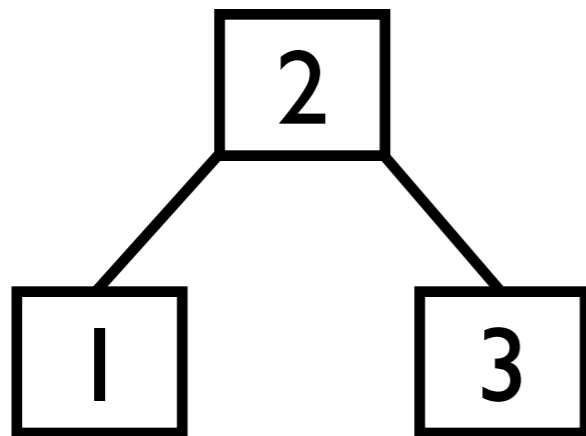


insert 4

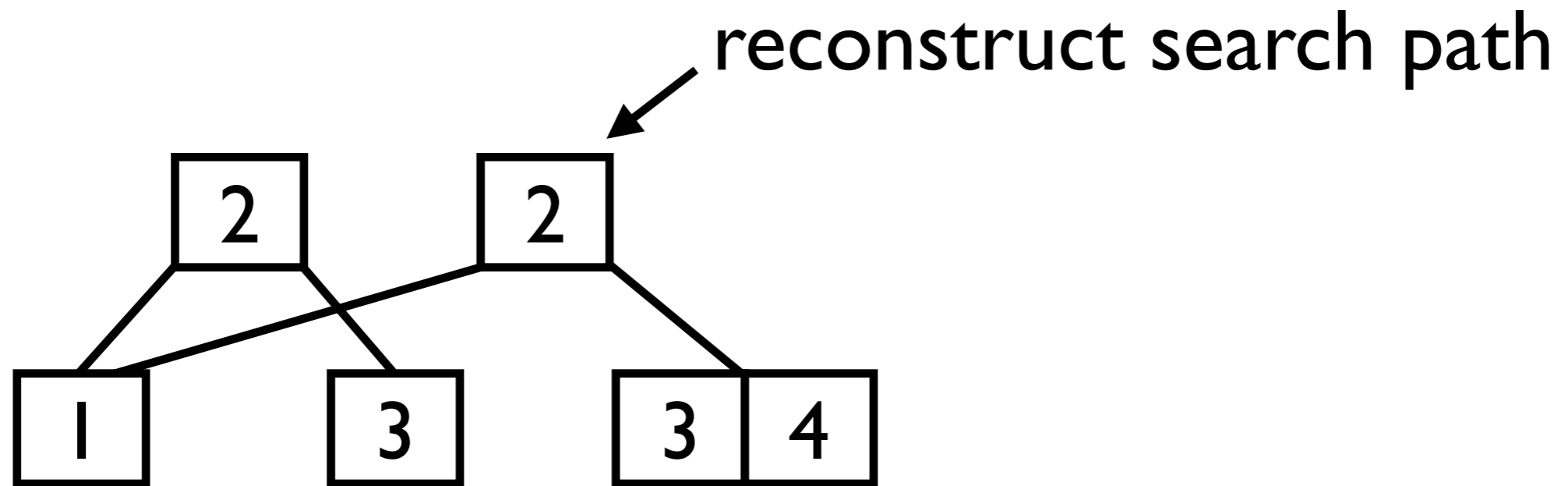
search



insert 4

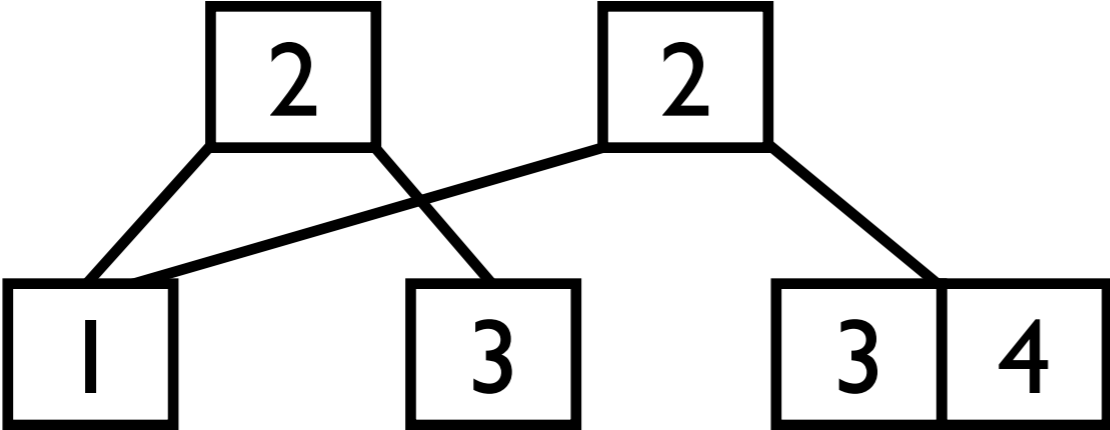


insert 4

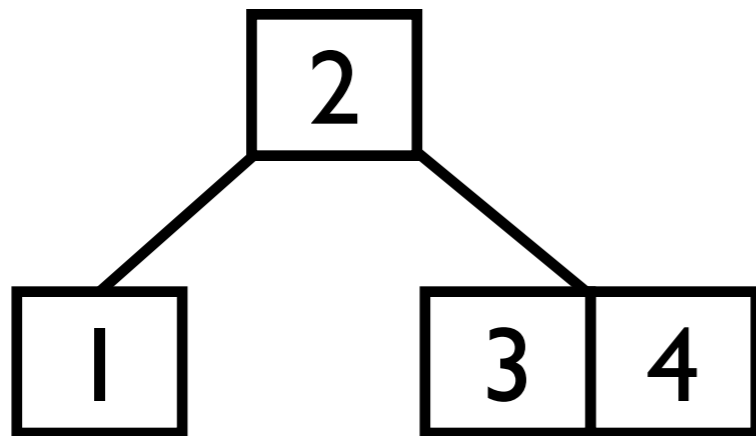


insert 4

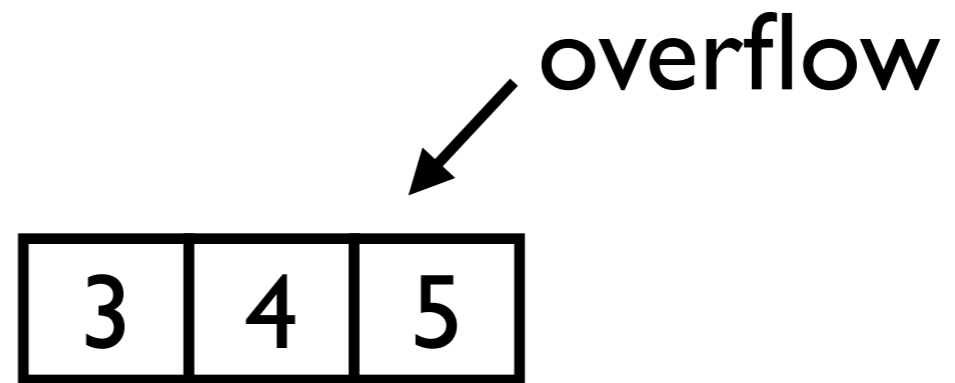
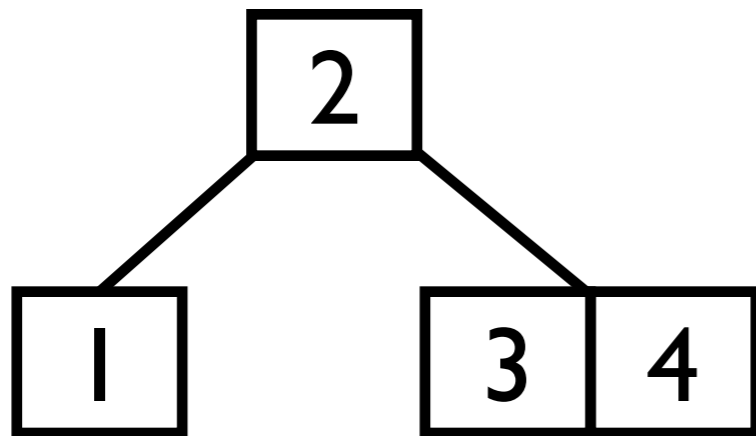
garbage-collect old version



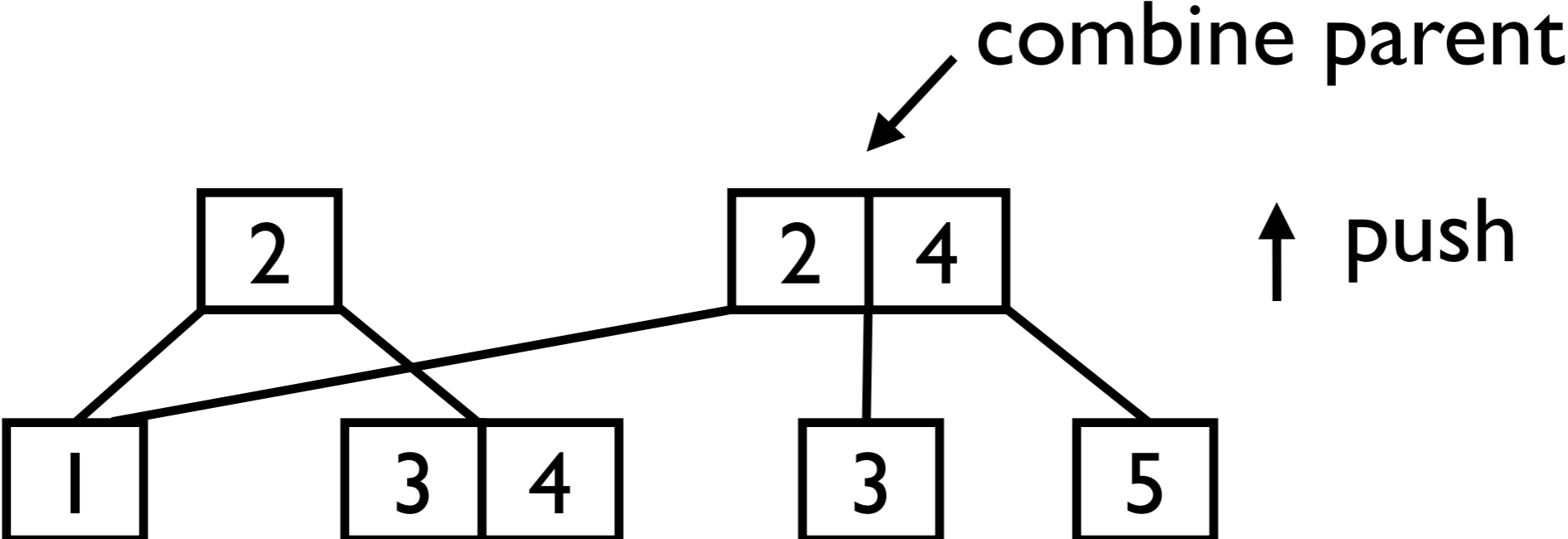
insert 4



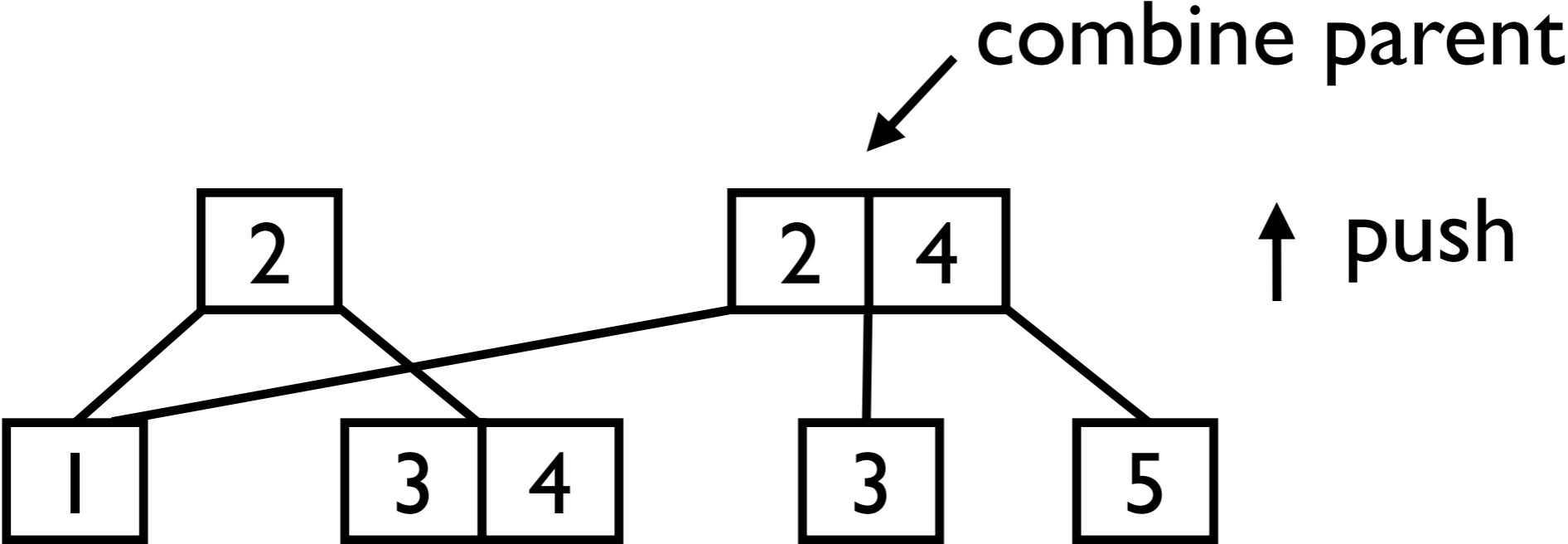
insert 5



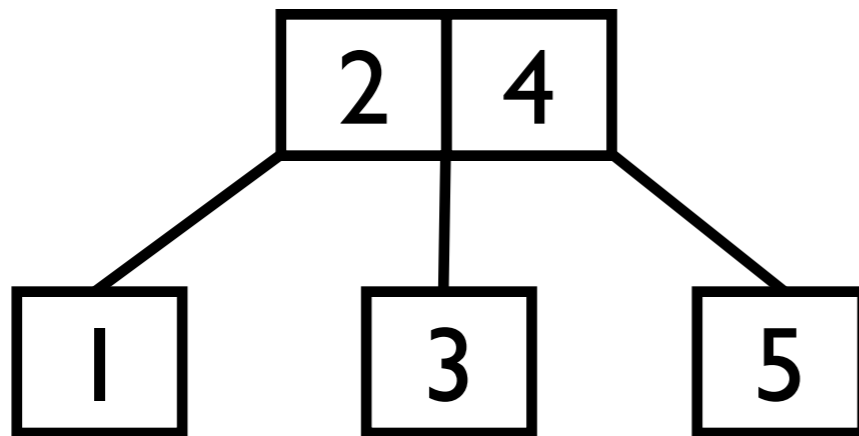
insert 5



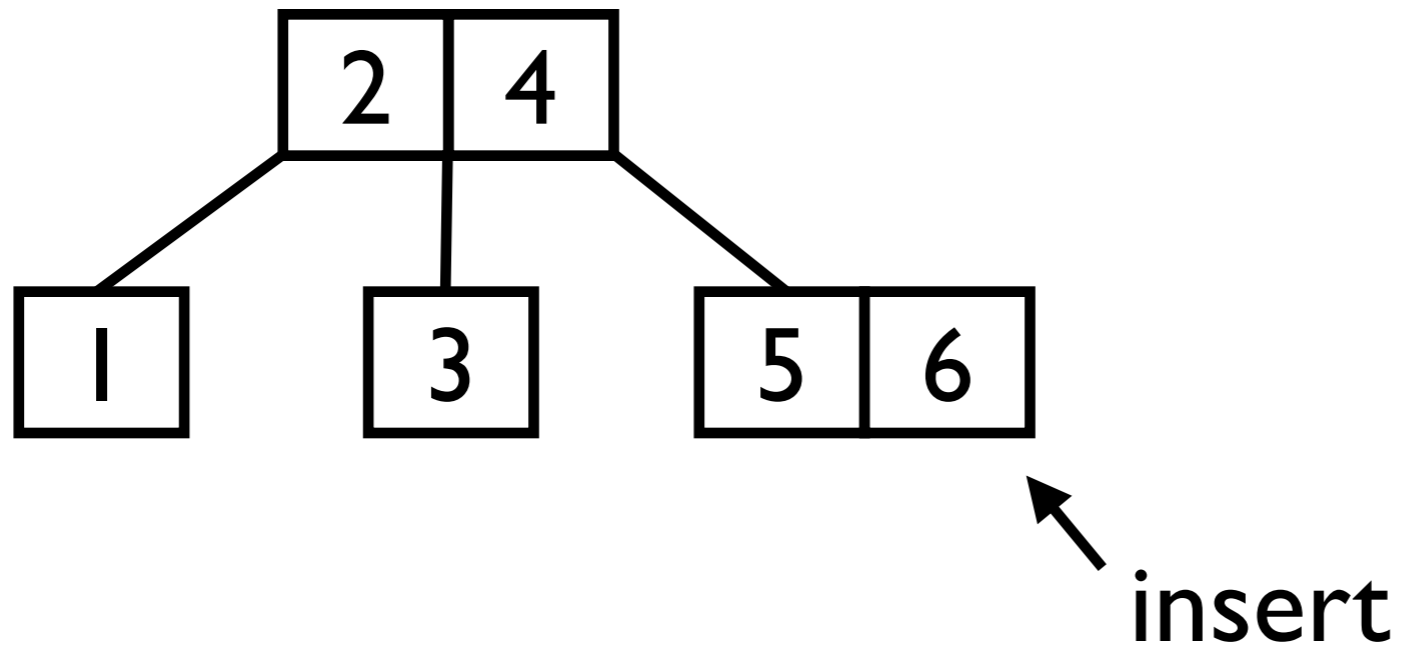
insert 5



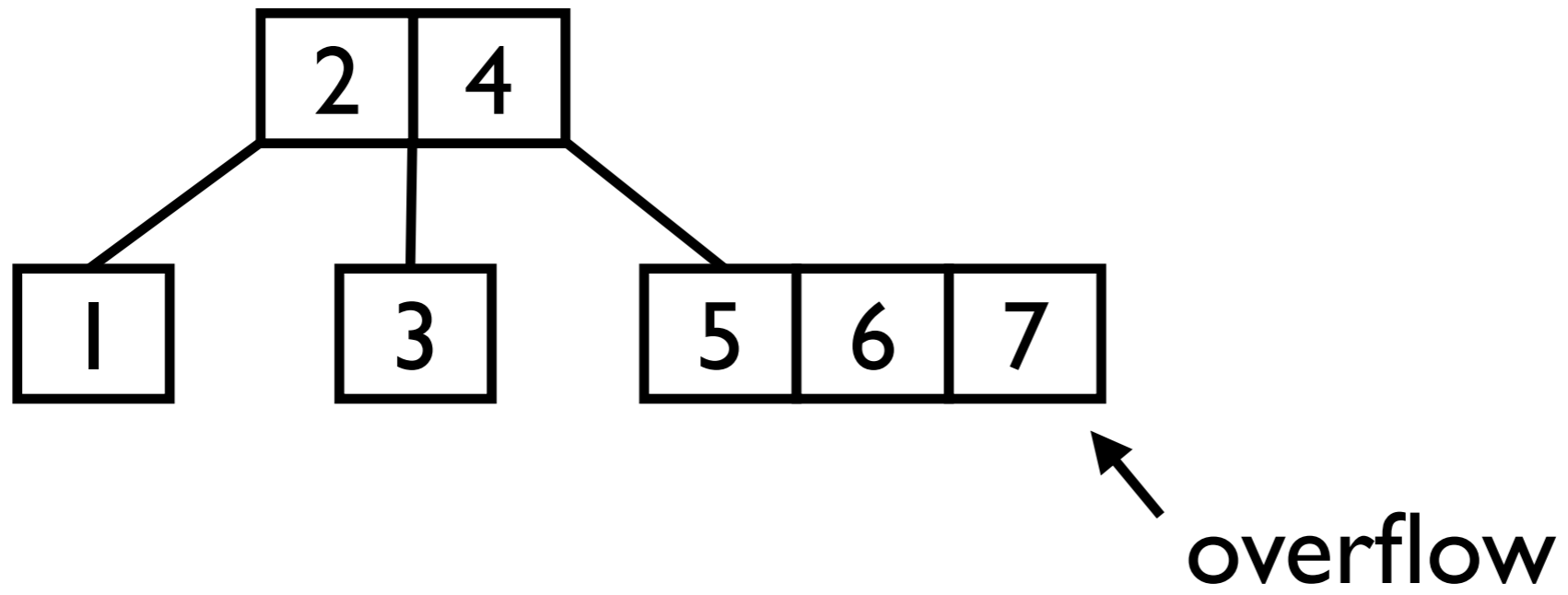
insert 5



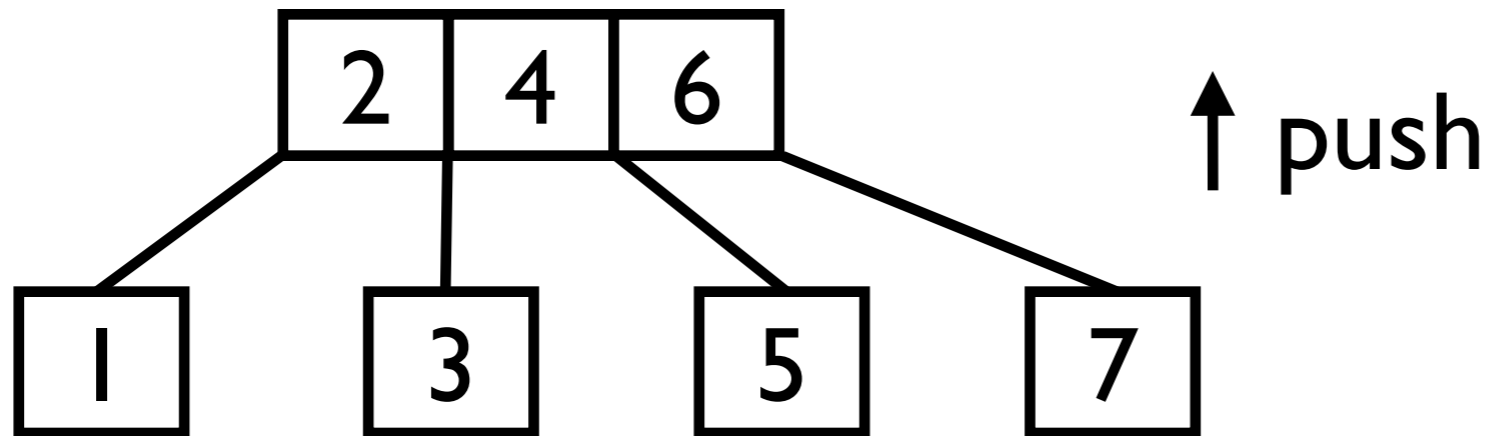
insert 6



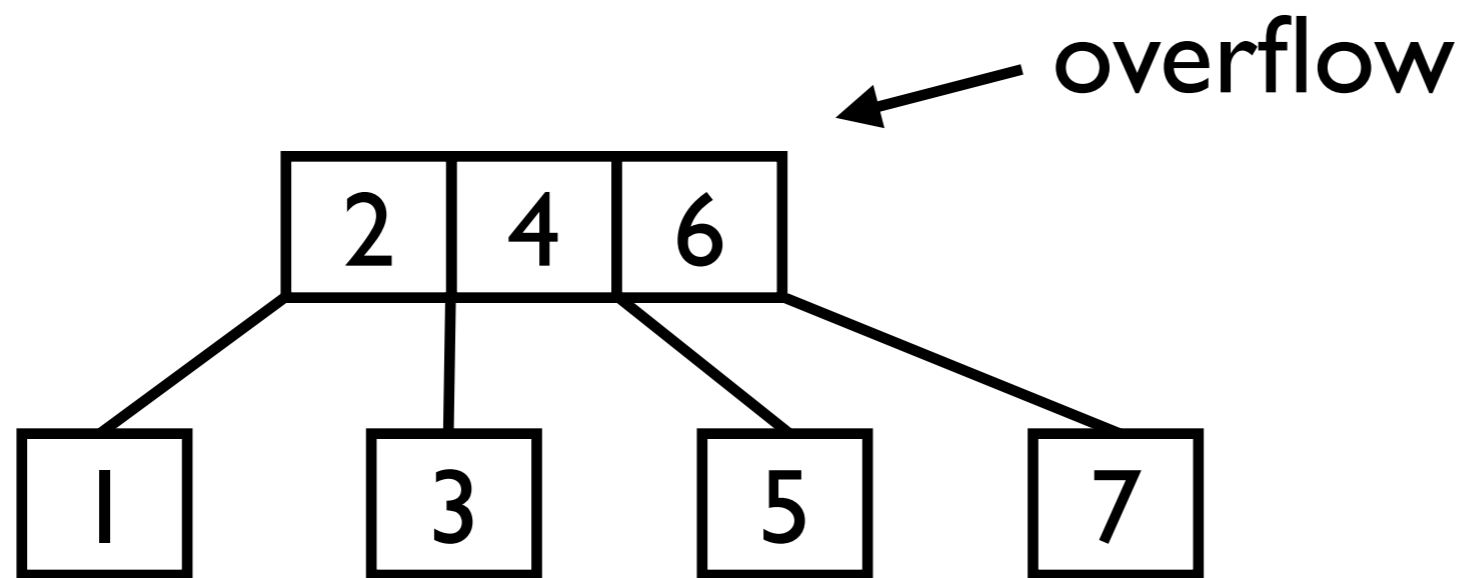
insert 7



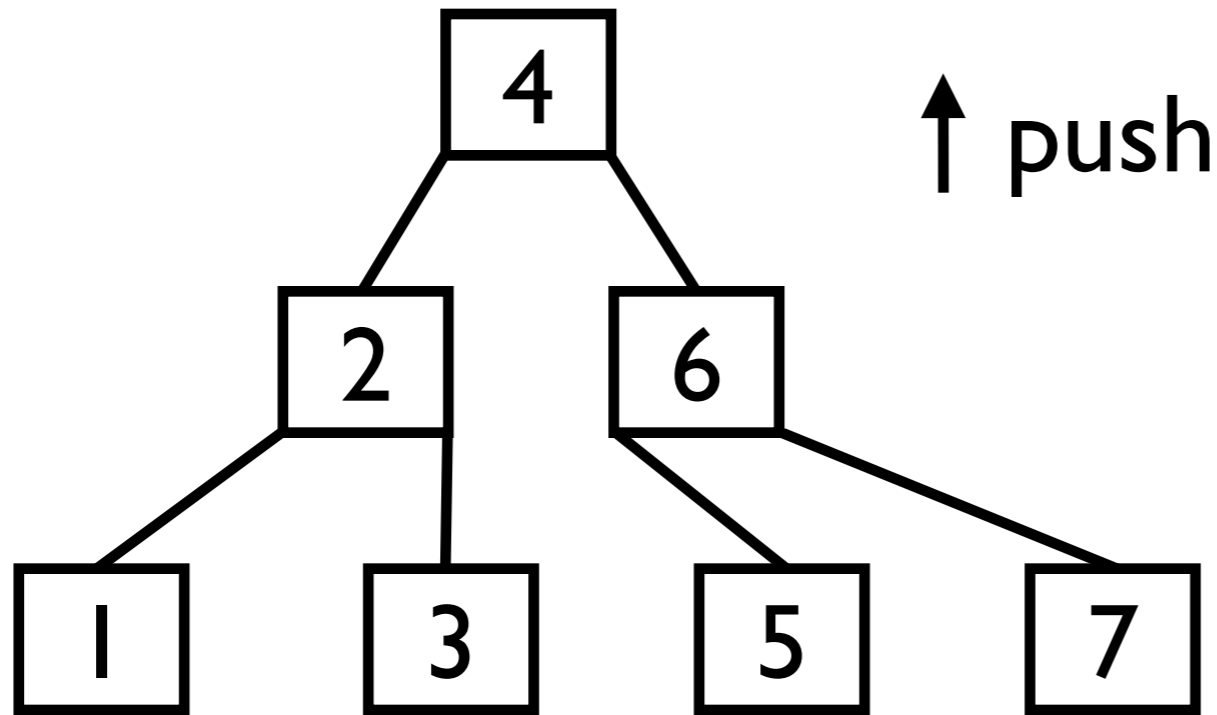
insert 7



insert 7



insert 7



data Nat = Z | S Nat

data T n a where

Br :: N n a → T (S n) a

LF :: T Z a

data N n a

= T1 (T n a) a (T n a)

| T2 (T n a) a (T n a) a (T n a)

data Tree a where

Tree :: T n a → Tree a

data T n a where

Br :: N n a \rightarrow T (S n) a

LF :: T Z a

data Tree a where

Tree :: T n a \rightarrow Tree a

insert :: Ord a \Rightarrow a \rightarrow Tree a \rightarrow Tree a

insert x (Tree t) = ins t

where

ins = undefined

data T n a where

Br :: N n a → T (S n) a

LF :: T Z a

data Tree a where

Tree :: T n a → Tree a

insert :: Ord a ⇒ a → Tree a → Tree a

insert x (Tree t) = ins t

where

ins :: Foo

ins = undefined

data T n a where

Br :: N n a → T (S n) a

LF :: T Z a

insert :: Ord a ⇒ a → Tree a → Tree a

insert x (Tree t) = ins t

where

ins :: Foo

ins = undefined

couldn't match expected type

'T n a → Tree a'

with actual type

'Foo'

data T n a where

Br :: N n a → T (S n) a

LF :: T Z a

insert :: Ord a ⇒ a → Tree a → Tree a

insert x (Tree t) = ins t

where

ins :: T n a → Tree a

ins = undefined

couldn't match expected type

'T n a → Tree a'

with actual type

'Foo'

data T n a where

Br :: N n a → T (S n) a

LF :: T Z a

insert :: Ord a ⇒ a → Tree a → Tree a

insert x (Tree t) = ins t

where

ins :: T n a → Ins n a

ins = undefined

data T n a where

Br :: N n a → T (S n) a

LF :: T Z a

insert :: Ord a ⇒ a → Tree a → Tree a

insert x (Tree t) = finish (ins t)

where

ins :: T n a → Ins n a

ins = undefined

finish :: Ins n a → Tree a

finish = undefined

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$
 $\text{insert } x (\text{Tree } t) = \text{finish } (\text{ins } t)$

where

$\text{ins} :: \text{T } n \ a \rightarrow \text{Ins } n \ a$

$\text{ins} = \text{undefined}$

$\text{finish} :: \text{Ins } n \ a \rightarrow \text{Tree } a$

$\text{finish} = \text{undefined}$

$\text{data } \text{Ins } n \ a$

$= \text{Keep } (\text{T } n \ a)$

$| \text{Push } (\text{T } n \ a) \ a \ (\text{T } n \ a)$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{finish } (\text{ins } t)$

where

$\text{ins} :: \text{T } n \ a \rightarrow \text{Ins } n \ a$

$\text{ins} = \text{undefined}$

$\text{finish} :: \text{Ins } n \ a \rightarrow \text{Tree } a$

$\text{finish } (\text{Keep } t) = \text{Tree } t$

$\text{finish } (\text{Push } a \ b \ c) = \text{Tree } (t1 \ a \ b \ c)$

$\text{data } \text{Ins } n \ a$

$= \text{Keep } (\text{T } n \ a)$

$| \text{Push } (\text{T } n \ a) \ a \ (\text{T } n \ a)$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{finish } (\text{ins } t)$

where

$\text{ins} :: \text{T } n a \rightarrow \text{Ins } n a$

$\text{ins} = \text{undefined}$

$\text{finish} :: \text{Ins } n a \rightarrow \text{Tree } a$

$\text{finish } (\text{Keep } t) = \text{Tree } t$

$\text{finish } (\text{Push } a b c) = \text{Tree } (t1 a b c)$

$\text{type Keep } t n a = \text{T } n a \rightarrow t$

$\text{type Push } t n a = \text{T } n a \rightarrow a \rightarrow \text{T } n a \rightarrow t$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{finish } (\text{ins } t)$

where

$\text{ins} :: \text{T } n a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$\text{ins} = \text{undefined}$

$\text{finish} :: \text{Ins } n a \rightarrow \text{Tree } a$

$\text{finish } (\text{Keep } t) = \text{Tree } t$

$\text{finish } (\text{Push } a b c) = \text{Tree } (t1 a b c)$

$\text{type Keep } t n a = \text{T } n a \rightarrow t$

$\text{type Push } t n a = \text{T } n a \rightarrow a \rightarrow \text{T } n a \rightarrow t$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$\text{ins} = \text{undefined}$

$\text{finish} :: \text{Ins } n a \rightarrow \text{Tree } a$

$\text{finish } (\text{Keep } t) = \text{Tree } t$

$\text{finish } (\text{Push } a b c) = \text{Tree } (t1 a b c)$

$\text{type Keep } t n a = \text{T } n a \rightarrow t$

$\text{type Push } t n a = \text{T } n a \rightarrow a \rightarrow \text{T } n a \rightarrow t$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$\text{ins} = \text{undefined}$

$\text{type Keep } t n a = \text{T } n a \rightarrow t$

$\text{type Push } t n a = \text{T } n a \rightarrow a \rightarrow \text{T } n a \rightarrow t$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n \ a \rightarrow \text{Keep } t \ n \ a \rightarrow \text{Push } t \ n \ a \rightarrow t$

$\text{ins} = \text{undefined}$

$\text{data } \text{T } n \ a \text{ where}$

$\text{Br} :: \text{N } n \ a \rightarrow \text{T } (\text{S } n) \ a$

$\text{LF} :: \text{T } Z \ a$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n \ a \rightarrow \text{Keep } t \ n \ a \rightarrow \text{Push } t \ n \ a \rightarrow t$

$\text{ins } \text{LF} = i$

where

$i \ \text{keep } \text{push} = \text{undefined}$

$\text{data } \text{T } n \ a \ \text{where}$

$\text{Br} :: \text{N } n \ a \rightarrow \text{T } (\text{S } n) \ a$

$\text{LF} :: \text{T } Z \ a$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$\text{ins } \text{LF} = i$

where

$i :: \text{Keep } t P a \rightarrow \text{Push } t P a \rightarrow t$

$i \text{ keep } \text{push} = \text{undefined}$

couldn't match type Z with P

$\text{data } \text{T } n a \text{ where}$

$\text{Br} :: \text{N } n a \rightarrow \text{T } (\text{S } n) a$

$\text{LF} :: \text{T } Z a$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$\text{ins } \text{LF} = i$

where

$i :: \text{Keep } t Z a \rightarrow \text{Push } t Z a \rightarrow t$

$i \text{ keep } \text{push} = \text{undefined}$

couldn't match type Z with P

$\text{data } \text{T } n a \text{ where}$

$\text{Br} :: \text{N } n a \rightarrow \text{T } (\text{S } n) a$

$\text{LF} :: \text{T } Z a$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n \ a \rightarrow \text{Keep } t \ n \ a \rightarrow \text{Push } t \ n \ a \rightarrow t$

$\text{ins } \text{LF} = i$

where

$i :: \text{Keep } t \ Z \ a \rightarrow \text{Push } t \ Z \ a \rightarrow t$

$i \ \text{keep} \ \text{push} = \text{undefined}$

where

$_ = \text{keep} :: \text{T } Z \ a \rightarrow t$

$_ = \text{push} :: \text{T } Z \ a \rightarrow a \rightarrow \text{T } Z \ a \rightarrow t$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n \ a \rightarrow \text{Keep } t \ n \ a \rightarrow \text{Push } t \ n \ a \rightarrow t$

$\text{ins } \text{LF} = i$

where

$i :: \text{Keep } t \ \text{Z } a \rightarrow \text{Push } t \ \text{Z } a \rightarrow t$

$i \ \text{keep } \text{push} = \text{keep } \text{LF}$

where

$_ = \text{keep} :: \text{T } \text{Z } a \rightarrow t$

$_ = \text{push} :: \text{T } \text{Z } a \rightarrow a \rightarrow \text{T } \text{Z } a \rightarrow t$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n \ a \rightarrow \text{Keep } t \ n \ a \rightarrow \text{Push } t \ n \ a \rightarrow t$

$\text{ins } \text{LF} = i$

where

$i :: \text{Keep } t \ \text{Z} \ a \rightarrow \text{Push } t \ \text{Z} \ a \rightarrow t$

$i \ \text{keep} \ \text{push} = \text{push } \text{LF} \ x \ \text{LF}$

where

$_ = \text{keep} :: \text{T } \text{Z} \ a \rightarrow t$

$_ = \text{push} :: \text{T } \text{Z} \ a \rightarrow a \rightarrow \text{T } \text{Z} \ a \rightarrow t$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$\text{ins } \text{LF} = \backslash \text{keep push} \rightarrow \text{push } \text{LF } x \text{ LF}$

where

$i :: \text{Keep } t \text{ Z } a \rightarrow \text{Push } t \text{ Z } a \rightarrow t$

$i \text{ keep push} = \text{push } \text{LF } x \text{ LF}$

where

$_ = \text{keep} :: \text{T } \text{Z } a \rightarrow t$

$_ = \text{push} :: \text{T } \text{Z } a \rightarrow a \rightarrow \text{T } \text{Z } a \rightarrow t$

$\text{insert} :: \text{Ord } a \Rightarrow a \rightarrow \text{Tree } a \rightarrow \text{Tree } a$

$\text{insert } x (\text{Tree } t) = \text{ins } t \text{ Tree } (((\text{Tree}.) .).t1)$

where

$\text{ins} :: \text{T } n \ a \rightarrow \text{Keep } t \ n \ a \rightarrow \text{Push } t \ n \ a \rightarrow t$

$\text{ins } \text{LF} = \backslash \text{keep } \text{push} \rightarrow \text{push } \text{LF} \ x \ \text{LF}$

where

$i :: \text{Keep } t \ \text{Z } a \rightarrow \text{Push } t \ \text{Z } a \rightarrow t$

$i \ \text{keep } \text{push} = \text{push } \text{LF} \ x \ \text{LF}$

where

$_ = \text{keep} :: \text{T } \text{Z } a \rightarrow t$

$_ = \text{push} :: \text{T } \text{Z } a \rightarrow a \rightarrow \text{T } \text{Z } a \rightarrow t$

$\text{ins} :: T\ n\ a \rightarrow \text{Keep}\ t\ n\ a \rightarrow \text{Push}\ t\ n\ a \rightarrow t$

$\text{ins}\ \text{LF} = \backslash\text{keep}\ \text{push} \rightarrow \text{push}\ \text{LF}\ x\ \text{LF}$

data $T\ n\ a$ where

$\text{Br} :: N\ n\ a \rightarrow T\ (S\ n)\ a$

$\text{LF} :: T\ Z\ a$

```
ins :: T n a → Keep t n a → Push t n a → t
ins (Br node) = i node
  where
    i = undefined
```

```
data T n a where
  Br :: N n a → T (S n) a
  LF :: T Z a
```

$\text{ins} :: \text{T n a} \rightarrow \text{Keep t n a} \rightarrow \text{Push t n a} \rightarrow \text{t}$

$\text{ins (Br node)} = \text{i node}$

where

$\text{i} :: \text{N P a} \rightarrow \text{Keep t M a} \rightarrow \text{Push t M a} \rightarrow \text{t}$

$\text{i} = \text{undefined}$

couldn't match type 'S P' with 'M'

data T n a where

$\text{Br} :: \text{N n a} \rightarrow \text{T (S n) a}$

$\text{LF} :: \text{T Z a}$

$\text{ins} :: \text{T n a} \rightarrow \text{Keep t n a} \rightarrow \text{Push t n a} \rightarrow \text{t}$

$\text{ins} (\text{Br node}) = \text{i node}$

where

$\text{i} :: \text{S p} \sim \text{m} \Rightarrow$

$\text{N p a} \rightarrow \text{Keep t m a} \rightarrow \text{Push t m a} \rightarrow \text{t}$

$\text{i} = \text{undefined}$

couldn't match type 'S P' with 'M'

data T n a where

$\text{Br} :: \text{N n a} \rightarrow \text{T (S n) a}$

$\text{LF} :: \text{T Z a}$

$ins :: T\ n\ a \rightarrow Keep\ t\ n\ a \rightarrow Push\ t\ n\ a \rightarrow t$

$ins\ (Br\ node) = i\ node$

where

$i :: S\ p\ \sim\ n \Rightarrow$

$N\ p\ a \rightarrow Keep\ t\ n\ a \rightarrow Push\ t\ n\ a \rightarrow t$

$i = \text{undefined}$

$\text{ins} :: \text{T n a} \rightarrow \text{Keep t n a} \rightarrow \text{Push t n a} \rightarrow \text{t}$

$\text{ins} (\text{Br node}) = \text{i node}$

where

$\text{i} :: \text{S p} \sim \text{n} \Rightarrow$

$\text{N p a} \rightarrow \text{Keep t n a} \rightarrow \text{Push t n a} \rightarrow \text{t}$

$\text{i} = \text{undefined}$

data N n a

$= \text{T1} (\text{T n a}) \text{ a} (\text{T n a})$

$| \text{T2} (\text{T n a}) \text{ a} (\text{T n a}) \text{ a} (\text{T n a})$

$\text{ins} :: \text{T n a} \rightarrow \text{Keep t n a} \rightarrow \text{Push t n a} \rightarrow \text{t}$

$\text{ins} (\text{Br node}) = \text{i node}$

where

$\text{i} :: \text{S p} \sim \text{n} \Rightarrow$

$\text{N p a} \rightarrow \text{Keep t n a} \rightarrow \text{Push t n a} \rightarrow \text{t}$

$\text{i} (\text{T2 a b c d e}) \text{ keep push}$

| $\text{x} == \text{b} = \text{undefined}$

| $\text{x} < \text{b} = \text{undefined}$

data N n a

$= \text{T1} (\text{T n a}) \text{ a} (\text{T n a})$

| $\text{T2} (\text{T n a}) \text{ a} (\text{T n a}) \text{ a} (\text{T n a})$

$\text{ins} :: T\ n\ a \rightarrow \text{Keep}\ t\ n\ a \rightarrow \text{Push}\ t\ n\ a \rightarrow t$

$\text{ins}\ (\text{Br}\ \text{node}) = i\ \text{node}$

where

$i :: S\ p\ \sim\ n \Rightarrow$

$N\ p\ a \rightarrow \text{Keep}\ t\ n\ a \rightarrow \text{Push}\ t\ n\ a \rightarrow t$

$i\ (\text{T2}\ a\ b\ c\ d\ e)\ \text{keep}\ \text{push}$

| $x == b = \text{undefined}$

| $x < b = \text{undefined}$

where

$_ = \text{keep} :: T\ n\ a \rightarrow t$

$_ = \text{push} :: T\ n\ a \rightarrow a \rightarrow T\ n\ a \rightarrow t$

$\text{ins} :: \text{T n a} \rightarrow \text{Keep t n a} \rightarrow \text{Push t n a} \rightarrow \text{t}$

$\text{ins} (\text{Br node}) = \text{i node}$

where

$\text{i} :: \text{S p} \sim \text{n} \Rightarrow$

$\text{N p a} \rightarrow \text{Keep t n a} \rightarrow \text{Push t n a} \rightarrow \text{t}$

$\text{i} (\text{T2 a b c d e}) \text{ keep push}$

| $\text{x} == \text{b} = \text{keep} (\text{t2 a x c d e})$

| $\text{x} < \text{b} = \text{undefined}$

where

$_ = \text{keep} :: \text{T n a} \rightarrow \text{t}$

$_ = \text{push} :: \text{T n a} \rightarrow \text{a} \rightarrow \text{T n a} \rightarrow \text{t}$

$i :: S p \sim n \Rightarrow$

$N p a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$i (T2 a b c d e) \text{ keep push}$

$| x == b = \text{keep } (t2 a x c d e)$

$| x < b = \text{undefined}$

where

$_ = \text{keep} :: T n a \rightarrow t$

$_ = \text{push} :: T n a \rightarrow a \rightarrow T n a \rightarrow t$

$i :: S p \sim n \Rightarrow N p a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$i (T2 a b c d e) \text{ keep push}$

$| x < b = \text{undefined}$

where

$_ = \text{keep} :: T n a \rightarrow t$

$_ = \text{push} :: T n a \rightarrow a \rightarrow T n a \rightarrow t$

$i :: S p \sim n \Rightarrow N p a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$i (T2 a b c d e) \text{ keep push}$

$| x < b = \text{ins } a \text{ rkeep rpush}$

where

$_ = \text{keep} :: T n a \rightarrow t$

$_ = \text{push} :: T n a \rightarrow a \rightarrow T n a \rightarrow t$

$\text{rkeep} :: T p a \rightarrow t$

$\text{rkeep } k = \text{undefined}$

$\text{rpush} :: T p a \rightarrow a \rightarrow T p a \rightarrow t$

$\text{rpush } p q r = \text{undefined}$

$i :: S p \sim n \Rightarrow N p a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$i (T2 a b c d e) \text{ keep push}$

$| x < b = \text{ins } a \text{ rkeep rpush}$

where

$_ = \text{keep} :: T n a \rightarrow t$

$_ = \text{push} :: T n a \rightarrow a \rightarrow T n a \rightarrow t$

$\text{rkeep} :: T p a \rightarrow t$

$\text{rkeep } k = \text{keep } (t2 k b c d e)$

$\text{rpush} :: T p a \rightarrow a \rightarrow T p a \rightarrow t$

$\text{rpush } p q r = \text{undefined}$

$i :: S p \sim n \Rightarrow N p a \rightarrow \text{Keep } t n a \rightarrow \text{Push } t n a \rightarrow t$

$i (T2 a b c d e) \text{ keep push}$

$| x < b = \text{ins } a \text{ rkeep rpush}$

where

$_ = \text{keep} :: T n a \rightarrow t$

$_ = \text{push} :: T n a \rightarrow a \rightarrow T n a \rightarrow t$

$\text{rkeep} :: T p a \rightarrow t$

$\text{rkeep } k = \text{keep } (t2 k b c d e)$

$\text{rpush} :: T p a \rightarrow a \rightarrow T p a \rightarrow t$

$\text{rpush } p q r =$

$\text{keep } (t1 (t1 p q r) b (t1 c d e))$

Could not deduce $(p \sim S p)$

$i :: S p \sim n \Rightarrow N p a \rightarrow Keep\ t\ n\ a \rightarrow Push\ t\ n\ a \rightarrow t$

$i (T2\ a\ b\ c\ d\ e)\ keep\ push$

$| x < b = ins\ a\ rkeep\ rpush$

where

$_ = keep :: T\ n\ a \rightarrow t$

$_ = push :: T\ n\ a \rightarrow a \rightarrow T\ n\ a \rightarrow t$

$rkeep :: T\ p\ a \rightarrow t$

$rkeep\ k = keep\ (t2\ k\ b\ c\ d\ e)$

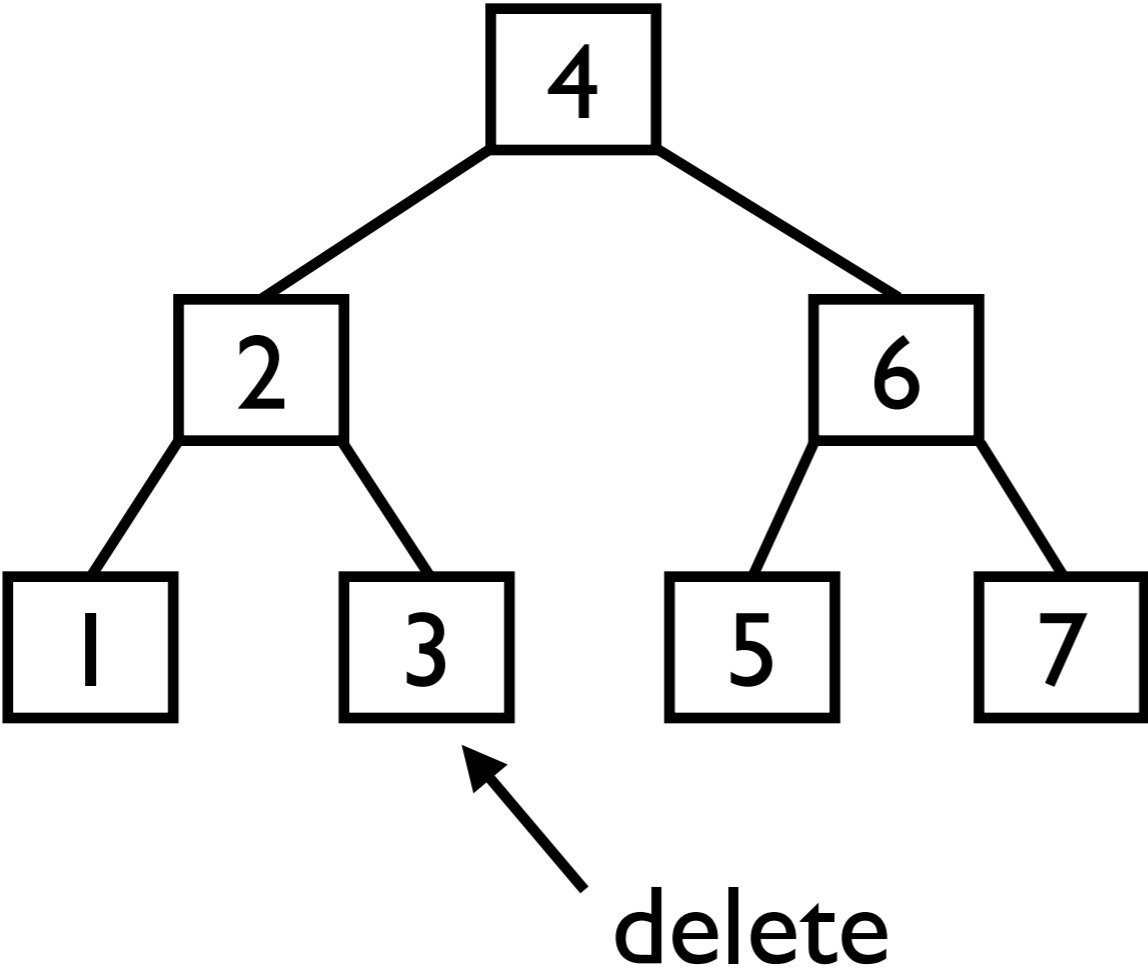
$rpush :: T\ p\ a \rightarrow a \rightarrow T\ p\ a \rightarrow t$

$rpush\ p\ q\ r =$

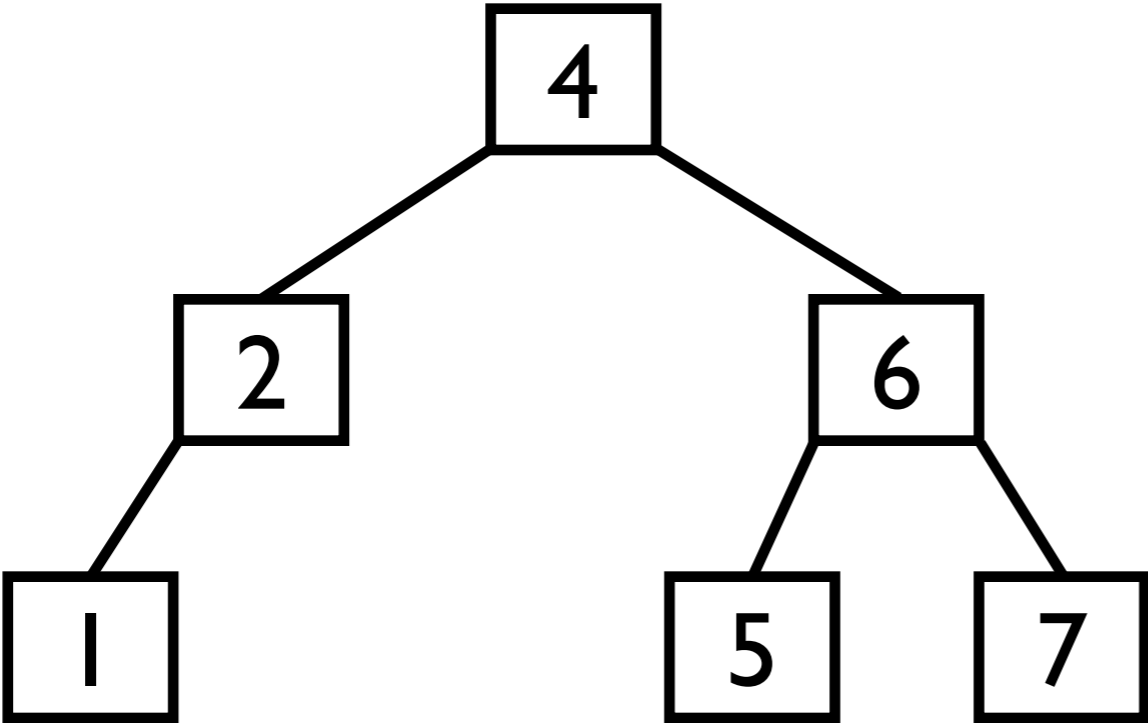
$push\ (t1\ p\ q\ r)\ b\ (t1\ c\ d\ e)$

Could not deduce $(p \sim S\ p)$

delete 3

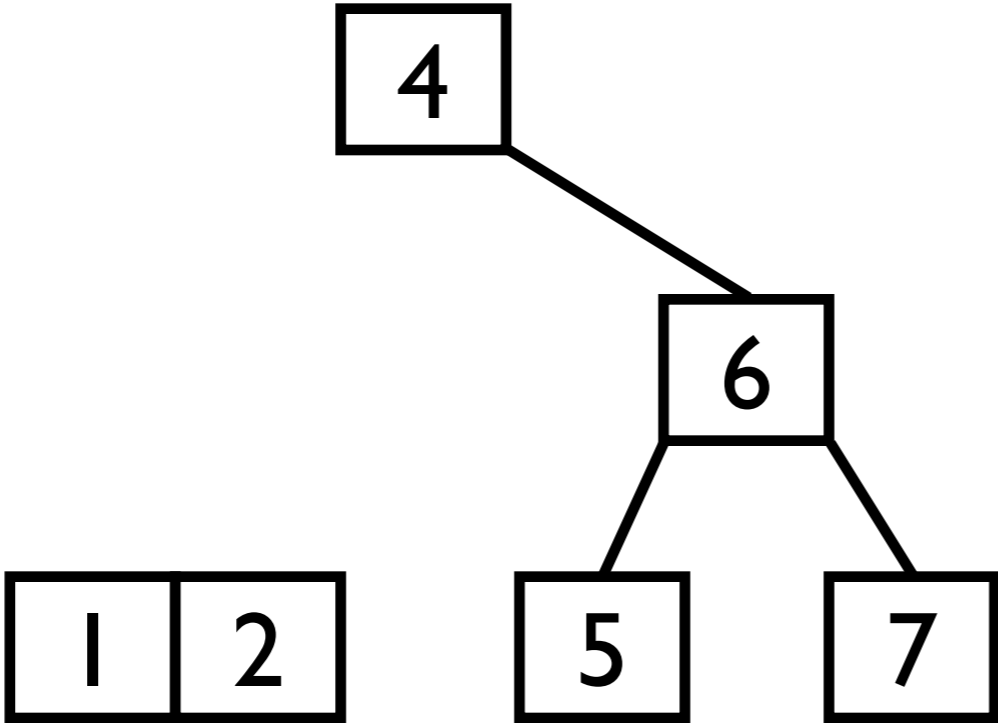


delete 3



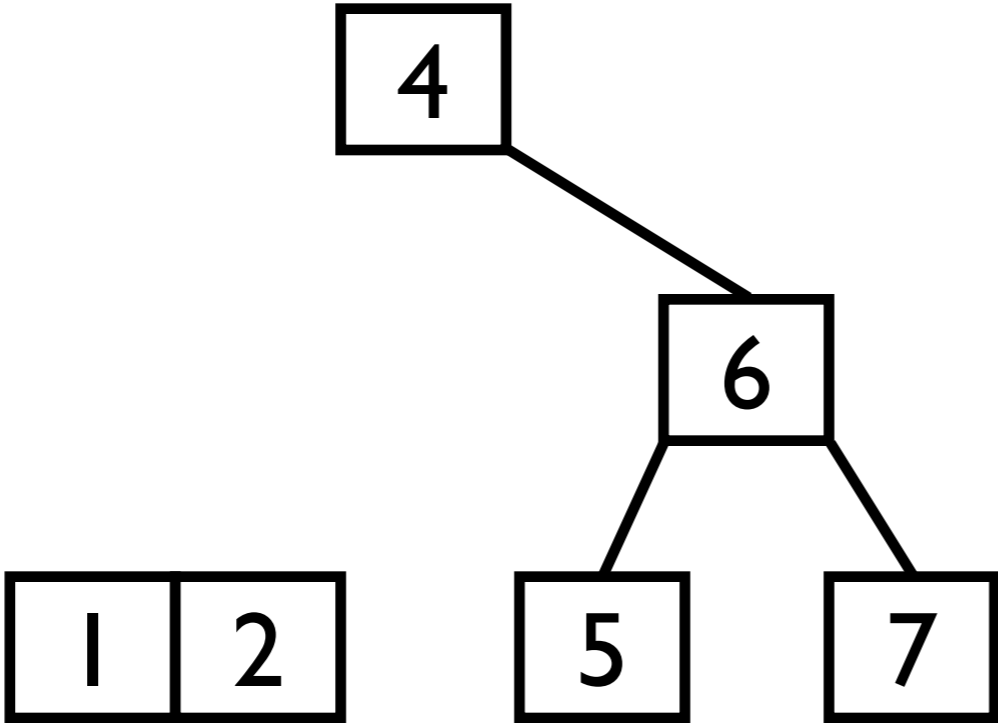
underflow

delete 3



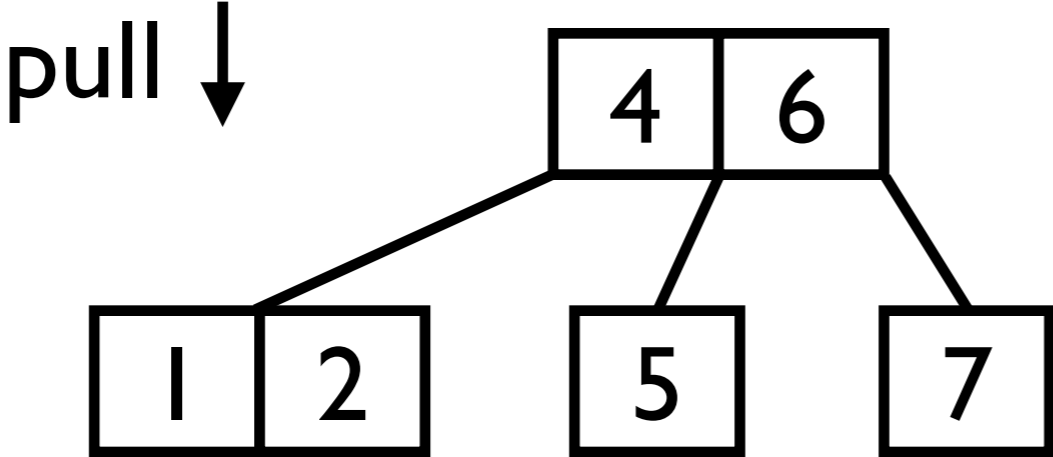
↓ pull

delete 3

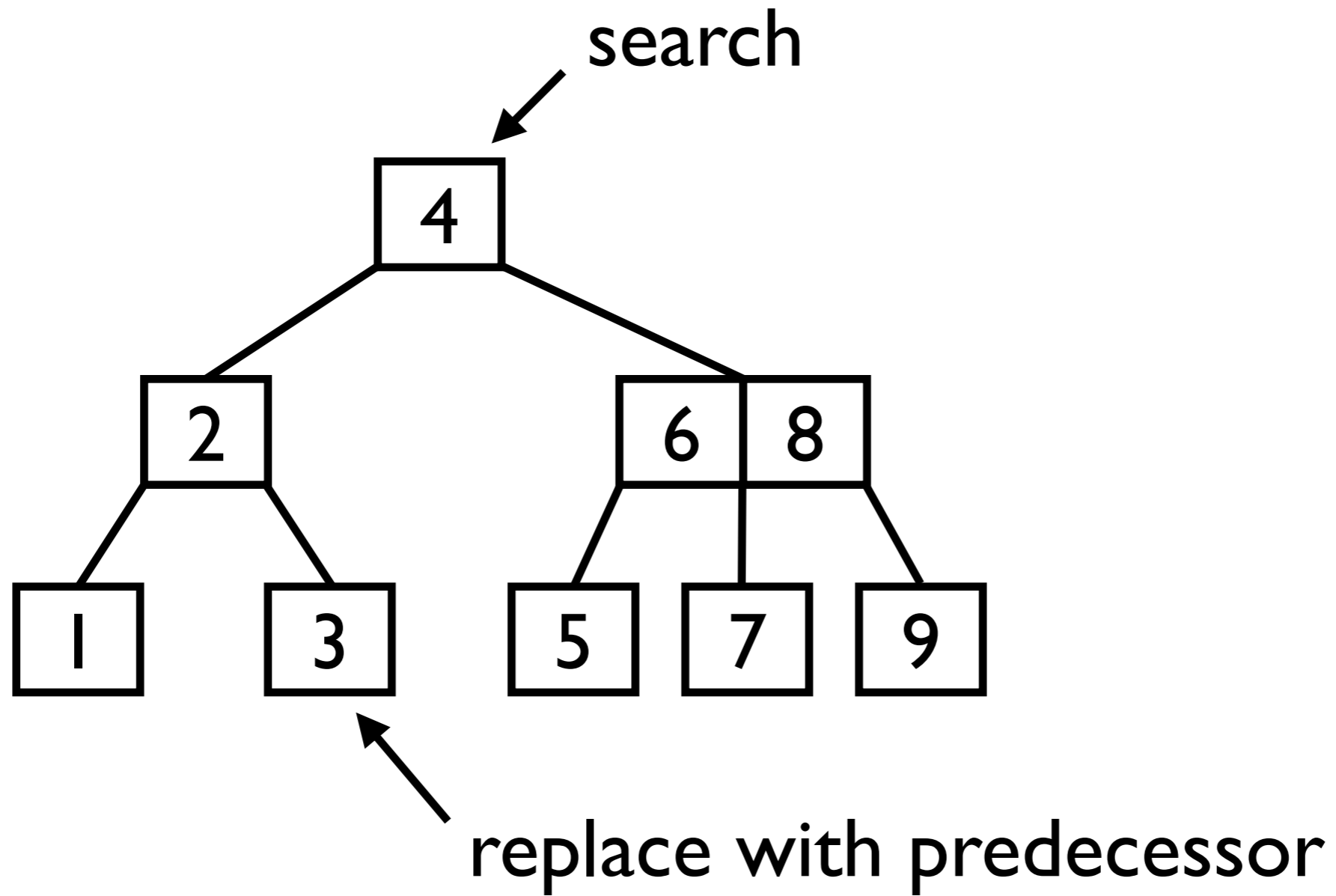


↓ pull

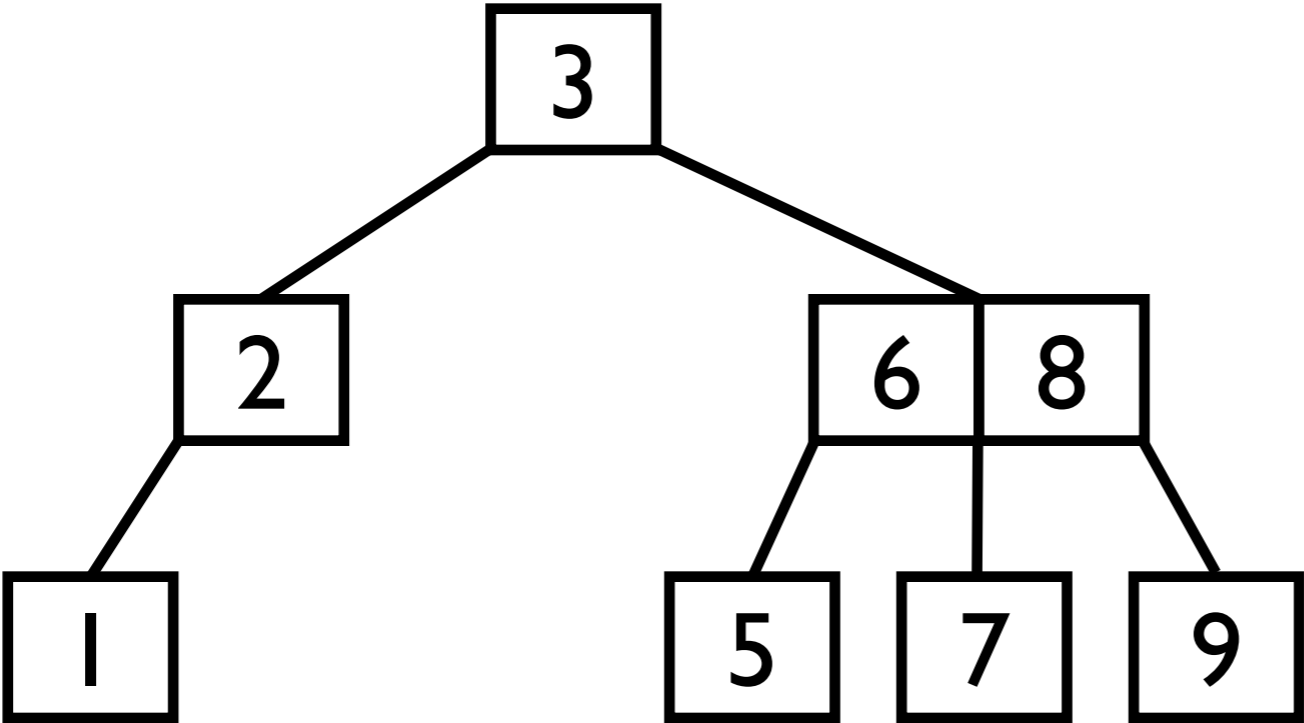
delete 3



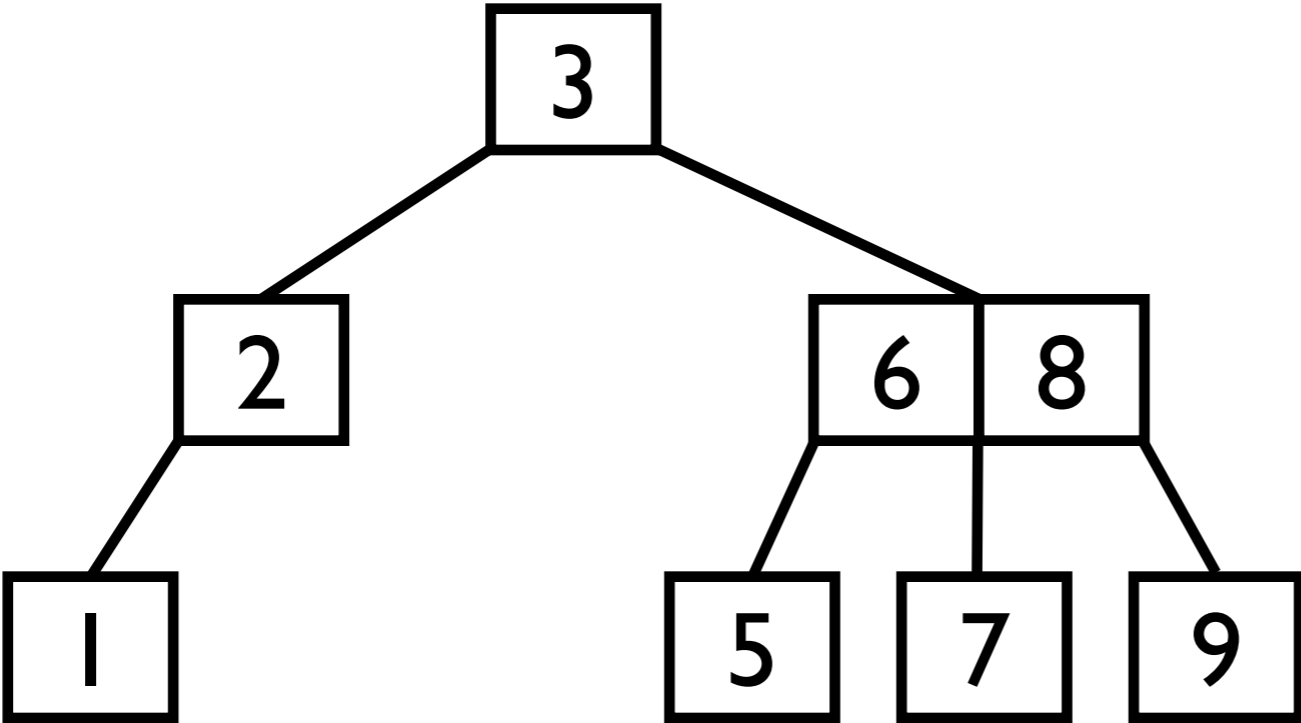
delete 4



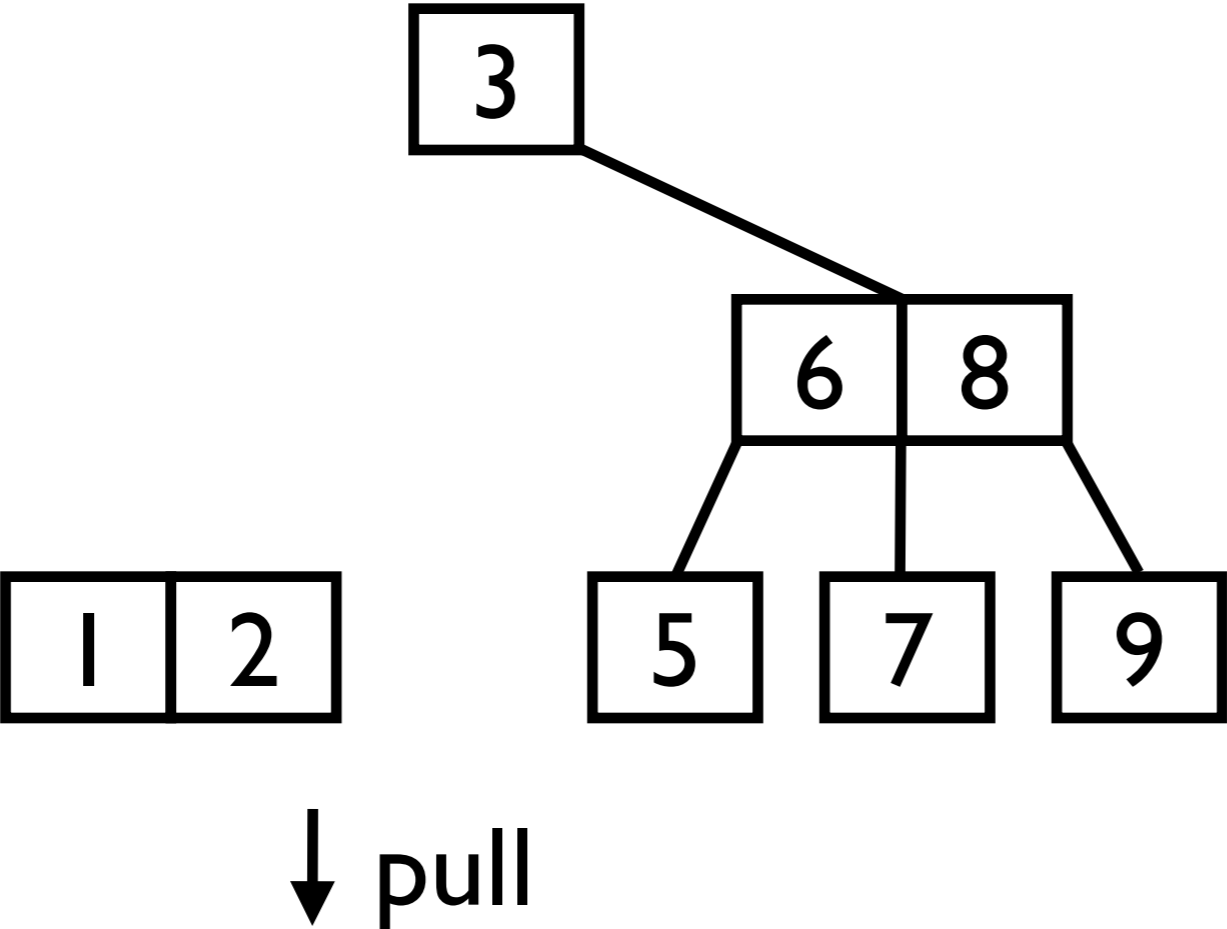
delete 4



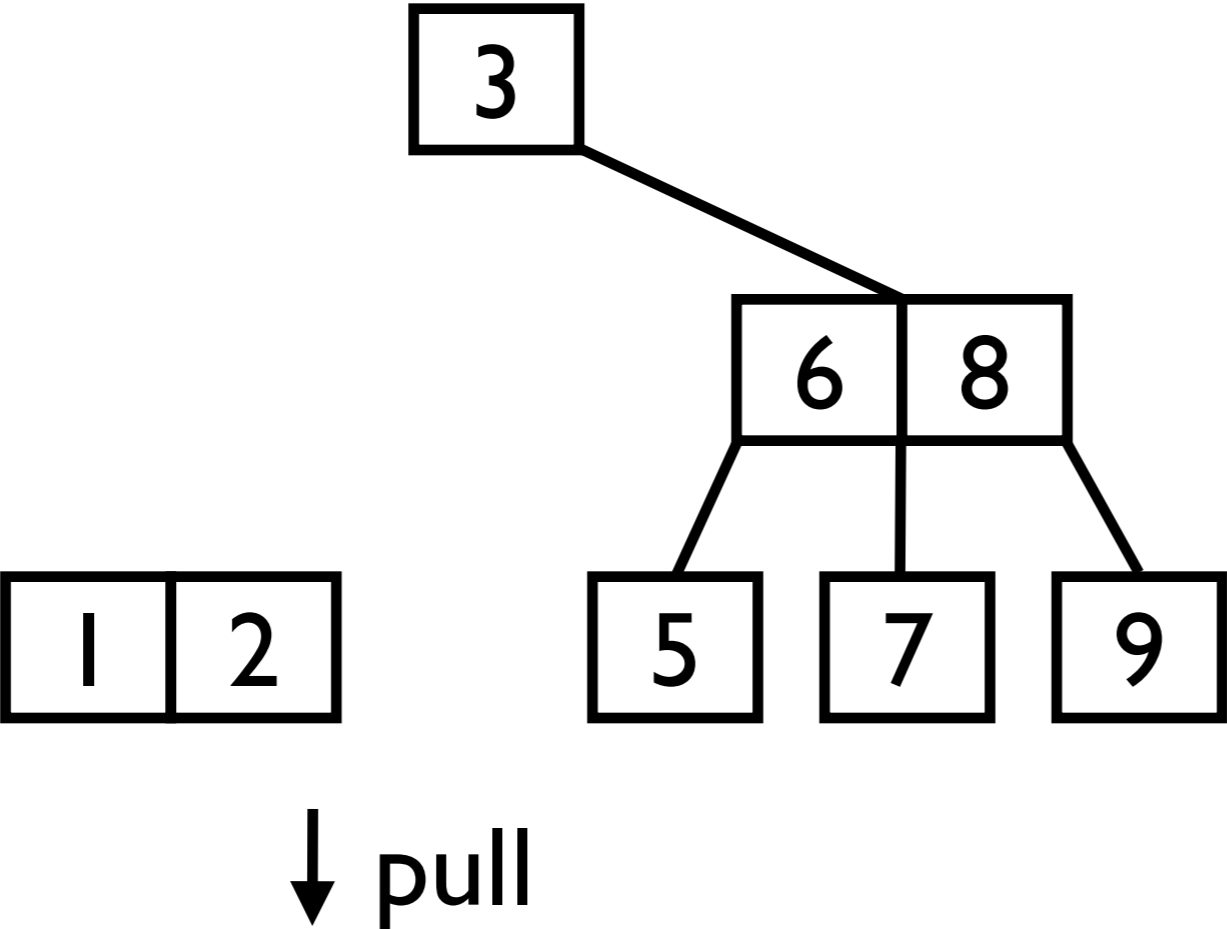
delete 4



delete 4



delete 4



delete 4

← rotate

